

downtick—a portion of the balance in the riskless asset is transferred to the risky asset. This rebalancing lowers the most extreme outcomes and lowers dispersion. It is the reduced volatility of rates, not diversification as traditionally defined, that generates diversification return.

But a closer look indicates that the phenomenon is arbitrary. Because this case involves combining a riskless asset with a risky asset, and because both assets have a geometric mean return equal to zero, the formula for the portfolio's diversification return reduces to a simple formula for the portfolio's geometric mean return.⁴ Substituting and solving with Equations (1, 2), and (3) with w as the weight of the risky asset and using the arithmetic mean (0.025) and variance (0.225²) from the example generates Equation (4) for the diversification return:

$$\begin{aligned} \text{Diversification return} \\ = \mu - [\sigma^2/2] = 0.025 w - 0.225^2 w^2/2 \end{aligned} \quad (4)$$

Equation (4) shows that the risky asset's arithmetic return adds to the diversification return and the risky asset's variance lowers the diversification return. As the risky asset's portfolio weight, w , increases, the portfolio's diversification return increases linearly, due to the risky asset's 2.5% arithmetic mean. But increases in w have a quadratic relationship in penalizing the diversification return for the risky asset's variance. As indicated in Exhibit 3, the diversification return (i.e., the geometric mean return) rises, reaches a maximum, and then declines as the risky asset's weight moves from zero to one.

Note that higher values of w always cause higher expected wealth, but that diversification return reaches a maximum at $w = \mu/\sigma^2$ and becomes negative beyond $w = 2\mu/\sigma^2$. The diversification return in Equation (4) is nothing more than a manifestation of the difference between the two assets' arithmetic mean returns and the reduction in the geometric mean caused by the higher volatility of assigning a higher portfolio weight to the risky asset. There is no reason to believe that the magnitude of the diversification return has any relevant economic meaning. Diversification return simply reflects the downward bias in a geometric mean return that is attributable to the concave relationship between prices and compounded yields. Diversification return does not generate higher expected wealth; it only reflects the concavity of the wealth transformation.

DIVERSIFICATION RETURN, PORTFOLIO REBALANCING, AND SERIAL CORRELATION

The effect of portfolio rebalancing on diversification return is an issue of importance. The effect of rebalancing on expected portfolio growth depends on the autocorrelation or serial correlation of the underlying assets. One substantial point of agreement among commentators on diversification return is that the active management process of rebalancing portfolios back towards some fixed weights is a mean-reverting strategy. Rebalancing generates higher expected portfolio value when asset prices mean-revert and lower expected portfolio value when asset prices trend. The explanation is simple: Rebalancing a portfolio to fixed weights involves selling assets that have experienced superior returns and buying assets that have experienced inferior returns. If the returns are mean-reverting (i.e., exhibit negative serial correlation), then rebalancing sells prior to relatively poor returns and buys prior to relatively high returns.

Advocates of the benefits of portfolio rebalancing in the context of diversification return do not require that asset returns be mean-reverting—they simply note that mean-reversion enhances diversification return. Willenbrock states "...the underlying source of diversification return is contained in the rebalancing. Rebalancing a portfolio involves selling assets that have appreciated in relative value and buying assets that have declined in relative value, as measured by their weights in the portfolio. This contrarian activity generates incremental returns as the assets fluctuate in value."

Exhibit 4 refutes the claim that portfolio rebalancing adds expected growth in wealth when the underlying assets are serially uncorrelated. We return to the case of assets experiencing two equally likely returns each year: +25% or -20%. Consider an equally weighted portfolio of two such assets with zero correlation that begins with \$1 and annually rebalances. Exhibit 4 depicts the 16 possible states over two periods for both a buy-and-hold strategy and an annually rebalanced strategy. The results are shown both in dollar values and compounded rates of return. The average terminal value of both strategies is the same: \$1.0506. This expected value reflects the arithmetic mean growth rate of 2.5% for each asset, compounded for two periods.

The average annualized realized rate of return is 1.874% for the rebalancing strategy and 1.867% for the

EXHIBIT 4

Equally Weighted Portfolios of Two Assets after Two Periods*

Asset Num.		Rebalanced		Buy and Hold		Differences	
1	2	Value	Rate	Value	Rate	Value	Rate
+	+	\$1.5625	25.00%	\$1.5625	25.00%		
+	+	\$1.2813	13.19%	\$1.2813	13.19%		
+	+	\$1.2813	13.19%	\$1.2813	13.19%		
+	+	\$1.0506	2.50%	\$1.1013	4.94%**	\$0.0506	2.44%
+	-	\$1.2813	13.19%	\$1.2813	13.19%		
+	-	\$1.0000	0.00%	\$1.0000	0.00%		
+	-	\$1.0506	2.50%	\$1.0000	0.00%***	\$0.0506	2.50%
+	-	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	+	\$1.2813	13.19%	\$1.2813	13.19%		
-	+	\$1.0506	2.50%	\$1.0000	0.00%***	\$0.0506	2.50%
-	+	\$1.0000	0.00%	\$1.0000	0.00%		
-	+	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	-	\$1.0506	2.50%	\$1.1013	4.94%**	\$0.0506	2.44%
-	-	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	-	\$0.8200	-9.45%	\$0.8200	-9.45%		
-	-	\$0.6400	-20.00%	\$0.6400	-20.00%		
Means:		\$1.0506	1.874%	\$1.0506	1.867%		

Note: The first four columns denote the return for the two periods, with each pair in chronological order and “+” denoting +25% and “-” denoting -20%.

*Each asset with equal probabilities of +25% and -20% returns and with no cross-sectional or serial correlations.

**Buy-and-hold strategy performs better, because assets “trended.”

***Rebalancing strategy performs better, because assets “mean reverted.”

buy-and-hold strategy. In the parlance of diversification return, the “added return” (which is really just an illusion from averaging rates) is from rebalancing. Exhibit 4 marks with double and triple asterisks the 4 of the 16 states that have different returns for the two strategies. The four states with different returns occur whenever the assets have different returns in both time periods. The difference in returns during the first time period causes the weights to differ going into the second period. The difference in the returns during the second time period causes the portfolios’ performance to differ.

The dollar differences between the strategies in each of the four states are equal to \$0.0506. However, the \$0.0506 differences form different percentages. The rebalanced strategy earns the incremental \$0.0506 when its dollar base is smaller (\$1.00), which causes the profit to be +2.50%. The buy-and-hold strategy earns the incremental \$0.0506 when its dollar base is higher (\$1.1013), which causes the profit to be only +2.44%.

Exhibit 4 demonstrates the equality of the expected values of rebalancing and buy-and-hold strategies with serially uncorrelated returns. Exhibit 4 also demonstrates that the expected geometric mean returns of the strate-

gies differ, not because of higher expected values, but rather from the averaging of rates. Diversification return is nothing more than a mirage in which the same dollar benefits appear larger, because they are calculated over different bases and then averaged.

Exhibit 4 illustrates an important point: An asset with a relatively high geometric mean return cannot be successfully arbitrated against an asset with a relatively low geometric mean returns, if both assets have the same arithmetic mean returns. An arbitrager that is long the rebalanced portfolio in Exhibit 4 and is short the buy-and-hold portfolio experiences two paths (marked ***) of \$0.0506 profit and two paths (marked **) of \$0.0506 losses. There is zero expected profit to an arbitrager from being long the higher average geometric mean strategy (rebalancing) and being short the lower average geometric mean strategy (buy and hold).

Exhibit 5 expands the same model used in Exhibit 4 by providing time horizons that vary from 1 period to 12 periods⁵ (and omitting the listing of the detailed outcomes). The expected value of the portfolio for each strategy grows at the same fixed compound growth rate of 2.5% per year. However, the geometric mean returns decline from the first period value of 2.50%

EXHIBIT 5

Equally Weighted Portfolios of Two Assets for Up to 12 Periods

Number of Periods	Rebalanced Portfolio			Buy-and-Hold Portfolio		
	Mean Ret.	E[Value]	Volatility	Mean Ret.	E[Value]	Volatility
1	2.50%	\$1.0250	0.184	2.50%	\$1.0250	0.184
2	1.87%	\$1.0506	0.117	1.87%	\$1.0506	0.118
3	1.66%	\$1.0769	0.093	1.64%	\$1.0769	0.094
4	1.56%	\$1.1038	0.080	1.52%	\$1.1038	0.082
5	1.50%	\$1.1314	0.072	1.45%	\$1.1314	0.073
6	1.45%	\$1.1597	0.065	1.40%	\$1.1597	0.067
7	1.42%	\$1.1887	0.061	1.36%	\$1.1887	0.062
8	1.40%	\$1.2184	0.057	1.32%	\$1.2184	0.058
9	1.38%	\$1.2489	0.053	1.29%	\$1.2489	0.055
10	1.37%	\$1.2801	0.051	1.27%	\$1.2801	0.053
11	1.36%	\$1.3121	0.048	1.25%	\$1.3121	0.050
12	1.35%	\$1.3499	0.046	1.23%	\$1.3499	0.048

Notes: Each asset has equal probabilities of +25% and -20% returns, with no cross-sectional or serial correlations. Mean return is the average geometric return, $E(\text{Value})$ is the portfolio's expected value from a starting value of \$1, and volatility is the standard deviation of the realized rates of return.

when the geometric mean return equals the arithmetic mean return toward 0.0% at the infinite time horizon. (Recall that the risky assets each have long-term geometric mean returns equal to 0.0%.)

Exhibit 5 displays means and volatilities of the annualized rates of return for both rebalancing and buy-and-hold strategies. First, note that the rebalanced strategy has higher expected geometric means for all time horizons beyond one year. Second, note that the rebalanced strategy has smaller dispersion in the realized growth rates. Rebalancing reduces wealth dispersion. Reduced wealth dispersion increases geometric mean returns. However, the expected value of each portfolio grows at the same compound rate: 2.5%.

Exhibits 4 and 5 explore the roll of rebalancing, using hypothetical data in which serial correlation is set to zero. Booth and Fama, Willenbrock, Bouchy et al., and Qian demonstrate that portfolio rebalancing generally resulted in improved geometric mean returns using actual market returns from various time periods, markets, and asset allocation levels. Part of these results can be explained by the illusion of geometric returns, but most is attributable to mean-reversion in the underlying market data, and even in the hypothetical examples the authors used. Consistently higher risk-adjusted growth in value from portfolio rebalancing in practice requires mean-reversion. Bouchy et al. reported that rebalancing worked for a two-stock portfolio (Apple and Starbucks) over the interval of 1994 to 2011, but they noted that the

strategy underperformed over the first four years, when apparently the returns were trending. Nevertheless, the reported robustness of the rebalancing strategy over various studies, using a variety of assets and time intervals based on actual market data, is impressive. These empirical studies attest to the effectiveness of rebalancing in markets such as equities when mean-reversion is prevailing, but err in attributing the source of enhanced value to diversification or diversification return.

Diversification return advocates argue that rebalancing creates diversification return through maintaining better diversification than is obtained using a buy-and-hold strategy. But rebalancing does not inherently keep a portfolio better diversified. Whether rebalancing a portfolio towards original weights increases or decreases diversification depends on the definition of diversification and on the original weights. In most equilibrium capital market theories, the most diversified portfolio is the market portfolio, because it contains no idiosyncratic risk. Generally, a market portfolio needs little or no rebalancing through time, because the weights naturally remain market weights (in the absence of differential dividend yields, share repurchases, or new offerings). Rebalancing a portfolio that began with market weights to maintain fixed weights (fixed at their original values) would tend to move the portfolio away from the market weights and therefore cause the portfolio to be less well diversified, according to most theories of diversification.

It is not clear that rebalancing makes a portfolio more or less diversified. Rebalancing restores the original weights after market forces drive some weights higher and some lower, but who is to say that the original weights provide better or worse diversification than the new weights? If a small company with a very small original weight soars in value, it is reasonable to believe that the portfolio can be better diversified by allowing its weight to increase. Similarly, allowing the very large weight of a large firm to fall when its value has fallen would likely improve diversification, relative to rebalancing. Thus, rebalancing can improve diversification in some cases and can increase idiosyncratic risk in other cases.

A clearer description of the effect of rebalancing is to describe the effect on return as “rebalancing return” and to note that rebalancing return should generally be positive when asset prices are mean-reverting and negative when asset prices are trending.

SUMMARY AND CONCLUSIONS

The expected compound rate of return is a misunderstood measure of performance, because it focuses on rates and creates an illusion that volatility “punishes” expected growth (not just growth rates). Bodie [1995] debunked the illusion of time diversification created by the use of averaged annual rates. Without serial correlation, it is clear that holding a risky asset for a longer period of time increases the dollar risk, even though it provides an illusion of reduced risk when the risk is measured by geometric mean returns. Our criticism of “diversification return” follows the same logic and criticizes the same type of illusory analysis. It is through the distorted lens of geometric mean analysis that reduced volatility of and by itself can be interpreted as generating higher expected portfolio value.

The illusion that diversification return generates incremental expected wealth gains comes from comparing a portfolio’s realized or expected geometric mean return to a flawed and meaningless benchmark: the weighted average of the portfolio’s assets geometric means. A similar illusion occurs in the perception that levered exchange-traded funds (ETFs) systematically experience price decay (i.e., expected losses), even when the underlying assets follow a Markov process.

The primary conclusions of our analysis are threefold:

1. It is true that portfolio rebalancing tends to generate higher geometric mean returns (i.e., diversification return), even when returns are serially uncorrelated. But the higher geometric mean returns do not cause higher expected portfolio values. Expected portfolio values are governed by arithmetic means, not geometric means or volatility.
2. Portfolio rebalancing tends to increase a portfolio’s expected value when asset prices are mean-reverting. This enhanced growth emanates from applying a mean-reverting strategy (i.e., rebalancing) to prices that are mean-reverting. The added expected portfolio value is not attributable to either reduced volatility or increased diversification.
3. The higher expected geometric mean of a low-volatility portfolio cannot be arbitrated against a high-volatility portfolio when both portfolios have the same arithmetic mean returns and when prices are Markov. Rebalancing generates arbitrage opportunities only when prices are mean-reverting.

The divergence of opinion with regard to the efficacy of diversification return originates from the difference between applying a rate-focused view and a value-focused view of expected growth. In other words, the controversy relates to whether an investor should be more concerned about expected long-term growth, expressed as an expected annualized rate or expressed as an expected total percentage change in value. The former is an arbitrary,⁶ nonlinear transformation of wealth; the latter is not.

Consider the following approximation, previously detailed: $g \approx \mu - [\sigma^2/2]$.

A rate-based approach views the geometric mean return, g , as the best measure of expected long-term growth and views the arithmetic mean, μ , as ignoring volatility. A value-based approach views the arithmetic mean return, μ , as the best measure of expected long-term growth and views the geometric mean, g , as being distorted downward by volatility.

Elton and Gruber [1974] carefully refute the optimality of a rate-based approach to selecting portfolios and conclude: “Portfolio decisions based on...the geometric mean of multi-period returns are often...inferior to decisions based on consideration of returns sequen-

tially over time... even when the distribution of returns is expected to be identical in each future period.” Simply put, wealth or some linear transformation of wealth serves as a better argument for a utility function than does a geometric mean.

In the rate-based view of diversification return, by holding the arithmetic mean constant and reducing the volatility of realized rates through portfolio rebalancing, an investor can increase a portfolio’s expected geometric mean return and therefore can create added averaged return through rebalancing. In the value-based view of diversification return, the arithmetic mean governs the expected value and volatility only plays a meaningful role in the context of risk aversion.

To ascertain whether a rate-based or value-based approach is better merely requires returning to the example of the bank offering 18-year CDs that offer either a guaranteed yield of 4% or a 50/50 chance of 0% and 8% yields. The rate-based view sees the equal chances of a 0% and 8% yield as having the same expected realized rates of return (i.e., geometric mean return). The value-based view sees the equal chances of a 0% and 8% yield as having a much higher expected value (roughly \$25,000) than the certain 4% yield (\$20,000). As advocates of the value-based view, we believe this can be arbitrated and will offer to borrow from anyone at a long-term fixed rate of return of 4% if they will allow us to lend to them at a 50/50 chance of receiving long-term compounded rates of either 0% or 8%.⁷

The CD example provides a clear analogy and a clear answer: Expected annualized rates are a deceptive measure of expected long-term growth. The rate-based approach is flawed, and although diversification return indicates increased expected annualized rates of return, it does not indicate increased expected value. However, portfolio rebalancing can serve as an effective mean-reverting strategy. When underlying returns are mean-reverting, rebalancing offers a free dessert. It does so through allocating away from previously high-performing assets toward previously low-performing assets, not through diversification or volatility reduction. The expected gains of rebalancing mean-reverting assets come from the expected losses of other traders who are implementing trending strategies, not from turning water into wine.

ENDNOTES

¹This process may be viewed as a CRR [Cox et al. 1979] binomial tree with riskless rate of 0.024693 and continuous volatility of 0.22314. The results of this article are not unique to these parameters or to the use of a binomial tree model. The results are driven by the concavity of the geometric mean return as a function of the total percentage change in wealth and therefore merely require dispersion in potential returns.

²To prove this point, consider a risky asset with expected return $E[r_m]$. Assuming serially uncorrelated returns, the asset’s expected total (non-annualized) return over T periods is $[1 + E[r_m]]^T - 1$. Specifically, $E[\prod(1 + r_{m,t})] = [1 + E[r_m]]^T$ because the expected values of each cross-product, $E[r_{m,t} \cdot r_{m,t-k}]$, is zero. Thus, all serially uncorrelated risky assets have multi-period expected total non-annualized returns directly related to their single-period expected returns and unrelated to their volatility. In other words, the single-period arithmetic mean return of serially uncorrelated assets correctly ranks the expected non-annualized total return of assets, but the geometric mean return does not, because geometric means depend on volatility.

³Latane’s [1959] pioneering work on using geometric mean return maximization as a portfolio optimization criterion merely claims that the strategy “falls within the generally accepted range of rational behavior” and that it is “a useful criterion.” Samuelson [1971] used a gambling analogy to criticize the optimization of the geometric mean return as a criterion for choice amongst risky ventures in his paper titled “The ‘Fallacy’ of Maximizing the Geometric Mean in Long Sequences of Investing or Gambling.” As Samuelson notes, “the novel criterion of maximizing the expected average compound return, which asymptotically leads to maximizing the geometric mean, is shown to be arbitrary.” A focus on geometric mean return may lead to a potentially useful tradeoff between risk and return, but it is an arbitrary tradeoff.

⁴Fernholz and Shay [1982] derive the same relationship (their Equation 20) in concluding that rebalancing “produces a constant accrual of revenues” that would “be absent in a passive portfolio.”

⁵The time horizon was not extended beyond 12 years, because a four-path tree with non-recombining nodes reaches 16,777,216 nodes after only 12 periods.

⁶The magnitude of the effect is driven by the time it takes the planet to orbit the sun.

⁷Based on a \$10,000 initial principal amount and a 10-year horizon, this wager would commit us to paying off the debt with roughly \$20,000 in 10 years, but it would allow us to receive a 50/50 chance of receiving \$10,000, or roughly \$40,000 at the same 10-year horizon.

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